Supplementary Materials: HACS4x: 4-Ply Helical Auxetic Capacitive Sensors for Strain Sensing E-textiles

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S1. Geometrical Model Details

This section serves to explain in more detail the mathematics of the geometrical model used in the main text and as a reference to the Maple code provided in the repository.

S1.1. Generation of a Helical Surface

The model begin with the parameters of the fibres: EAF initial diameter $D_{E,i}$ and Poisson's ratio ν, IEF diameter D_I, initial helical pitch of the system p_i, and number of turns n. The fibres in 3D space are modelled starting from the equation for a 3D helical path concentric with the x-axis parameterized by $t = 0, \ldots, 2\pi$:

$$\vec{h}(t) = \begin{bmatrix} \frac{p}{2\pi} & r_M \cos(t + \theta) & r_M \sin(t + \theta) \end{bmatrix}, \tag{S1}$$

with major radius r_M, pitch p, and phase offset θ (shown in Fig. S1a). A circle of minor radius r_m parameterized by u is revolved along the tangent of the path to obtain a helical surface (Fig. S1c):

$$\vec{H}(t, u) = \vec{h}(t) + r_M \cos(t + \theta)\vec{r}(t) + r_m \sin(u)\vec{s}(t), \tag{S2}$$

where the orthonormal bases $\vec{q}(t)$, $\vec{r}(t)$, and $\vec{s}(t)$ (Fig. S1b)) are the normalized tangent vector to $\hat{h}$, the orthogonal vector pointing toward the x-axis, and $\vec{q}(t) \times \vec{r}(t)$, respectively. At strain $\varepsilon = 0$, the EAFs are modelled as helices with $r_M = r_m = \frac{D_{E,i}}{2}$, $p = p_i$, and with $\theta = 0$ for one EAF and $\theta = \pi$ for the other. Equation (3) shows the full equation for a helical surface.

$$\vec{H}(t, u) = \begin{cases} x(t, u) = \frac{p}{2\pi}t + \frac{r_M r_m \sin(u)}{\sqrt{r_M^2 + \left(\frac{p}{2\pi}\right)^2}} \\ y(t, u) = r_M \cos(t + \theta) - r_m \cos(t + \theta)\cos(u) + \frac{p \cdot r_m \sin(t + \theta)\sin(u)}{2\pi\sqrt{r_M^2 + \left(\frac{p}{2\pi}\right)^2}} \\ z(t, u) = r_M \sin(t + \theta) - r_m \sin(t + \theta)\cos(u) - \frac{p \cdot r_m \cos(t + \theta)\sin(u)}{2\pi\sqrt{r_M^2 + \left(\frac{p}{2\pi}\right)^2}} \end{cases} \tag{S3}$$

S1.2. Position of the IEFs

Each EAF in the 3D double helix may be sliced by a plane normal to the x-axis by solving the x-component of $\vec{H}(t, u)$ for t and substituting it back into the other components to obtain a closed curve in the cross-section $\vec{C}(u)$. From the cross section, the position of the
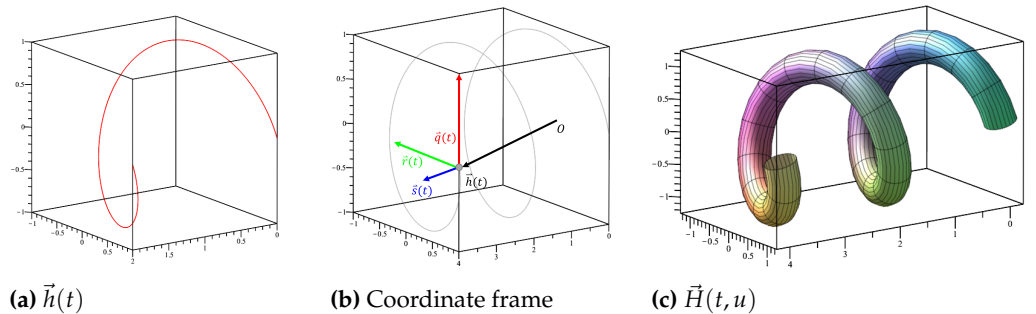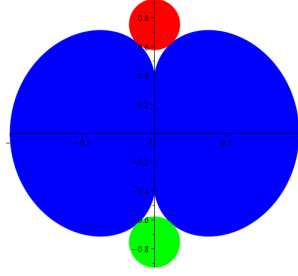

(a) $\vec{h}(t)$　　　　　　**(b)** Coordinate frame　　　　　　**(c)** $\vec{H}(t, u)$

Figure S1. Generation of a helical surface.

Figure S2. Position of the IEF (red, green) in the helical EAF complex (blue) grooves.

IEFs is determined by finding the closest that they can be positioned in the groove to have point contacts with the EAFs, corresponding to the smallest distance to the origin along the z-axis that a circular cross section can be fit. Geometrically, this is done by finding the value of u such that its distance to the z-axis along a vector perpendicular to the EAF curve is equal to the IEF diameter. The perpendicular direction to $\vec{C}(u)$ is:

$$\vec{C}_{\perp}(u) = \left[-\frac{d\vec{C}_z(u)}{du} \quad \frac{d\vec{C}_y(u)}{du} \right]. \tag{S4}$$

From Equation (4), the value of u is found that satisfies the condition:

$$\frac{-\vec{C}_y(u)}{\cos\left(\tan^{-1}\left(\frac{\vec{C}_{\perp,z}(u)}{\vec{C}_{\perp,y}(u)} \right) \right)} - \frac{D_I}{2} = 0. \tag{S5}$$

The helical surfaces corresponding to the IEFs have r_M equal to the z-coordinate from the solution of Equation (5), $r_m = D_I$, and $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, resulting in the initial cross-sectional geometry shown in Fig. S2.

S1.3. Behaviour of IEFs During Strain

The IEF helices' minor radius is constant and their major radii are only dependent on initial conditions and strain ε. Equation (6) shows an equality to solve for $r_{M,I}$ of the IEF helices as a function of ε, by equating their linear length l_I, which remains constant.

$$l_I(0) = l_I(\varepsilon)$$
$$n\sqrt{(\pi(D_I + D_E(0)))^2 + p^2} = n\sqrt{(\pi(2r_{M,I}(\varepsilon)))^2 + (p(1+\varepsilon))^2}$$
$$r_{M,I}(\varepsilon) = \frac{1}{2\pi}\sqrt{\pi^2(D_E(0) + D_I(0))^2 - p^2\varepsilon(\varepsilon+2)} \tag{S6}$$

Using the model assumptions described in the main text, $r_{M,I}(\varepsilon)$ from Equation (6) is sufficient to calculate capacitance between the w ires. The geometry of the EAFs during strain is not necessary, but described below to allow for future improvements to the model (such as taking into account the dielectric constants of air vs. EAF material).

S1.4. Behaviour of EAFs During Strain

During straining, the shape of the elastic EAFs is constrained: (i) by the length of the helical path (Equation (7)), and (ii) by their linear extension from Poisson's ratio (Equation (8)). Both of these conditions are dependent on D_E as it evolves with strain.

$$l_{E,h}(\varepsilon) = n\sqrt{4\pi^2\left(\frac{D_E(\varepsilon) + D_I}{2} - \frac{1}{2\pi}\sqrt{(D_E(0) + D_I)^2\pi^2 - p^2\varepsilon(\varepsilon+2)} \right)^2 + p^2(1+\varepsilon)^2} \tag{S7}$$
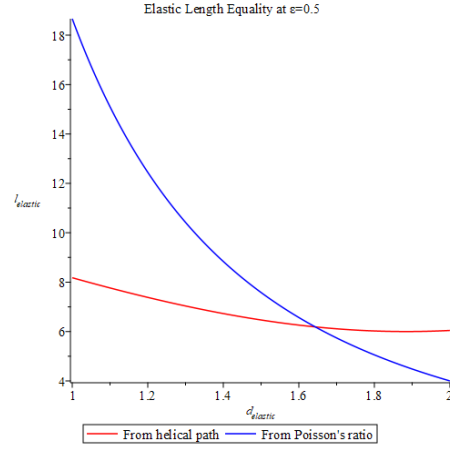
Figure S3. Solving the equality from Equations (7) and (8) to determine the EAF parameters at any strain.

$$l_{E,\nu}(\varepsilon) = \exp\left(-\frac{\ln\left(\frac{D_E(\varepsilon)}{D_E(0)}\right)}{\nu}\right) l_{E,0} \tag{S8}$$

By solving the equality $l_{E,h}(\varepsilon) = l_{E,\nu}(\varepsilon)$, the EAF diameter $D_E(\varepsilon)$ and thus length may be found. This was done numerically in practice, visualized in Fig. S3.

S1.5. Calculation of Capacitance

Given the evolution of the IEF major radius $r_{M,I}(\varepsilon)$ with strain, the formula for capacitance of two parallel wires (Equation (9)) may be used to estimate the capacitance of the sensor.

$$C(\varepsilon) = \frac{\pi \epsilon_0 \epsilon_r l_I(\varepsilon)}{\ln\left(\frac{2r_{M,I}(\varepsilon)}{D_I} + \sqrt{\frac{4r_{M,I}(\varepsilon)}{D_I} - 1}\right)} \tag{S9}$$